Math 005C — Exam 2 Study Guide

This study guide is designed to help you prepare specifically for Exam 2. Every item below corresponds directly to skills or ideas that appear on the exam.

1. Limits of Multivariable Functions

What you should be able to do:

- Evaluate limits of two-variable functions by direct substitution when the function is continuous at the point.
- Test whether a limit exists by approaching along two or more different paths.
- State a conclusion about a limit's existence based on path analysis.

You should understand:

- Why direct substitution works only when the function is defined and continuous at the target point.
- How the two-path test establishes that a limit does not exist: if two paths yield different values, the limit cannot exist.
- Why agreement along two paths is not sufficient to prove a limit exists—only disagreement proves non-existence.

2. Partial Derivatives

What you should be able to do:

- Compute first-order partial derivatives f_x and f_y by treating the other variable as a constant.
- Compute second-order partial derivatives f_{xx} and the mixed partial f_{xy}.

You should understand:

- The meaning of a partial derivative as the rate of change of f in one coordinate direction while all other variables are held fixed.
- Clairaut's Theorem: for "nice" functions, mixed partials are equal ($f_{xy} = f_{yx}$), so the order of differentiation does not matter.
- How to systematically apply product, chain, and power rules within a partial derivative computation.

3. Directional Derivatives and the Gradient

What you should be able to do:

- Compute the gradient vector ∇f at a given point.
- Find the directional derivative $D_{\mathbf{u}} f$ in a specified direction by first constructing a unit vector.
- Identify the direction of steepest ascent and the maximum rate of change at a point.
- Determine whether a function is increasing or decreasing in a given direction using the sign of the directional derivative.

You should understand:

- The gradient ∇f always points in the direction of greatest increase of f, and its magnitude equals the maximum rate of change.
- The directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ is a dot product—size matters, so you must normalize the direction vector before computing.
- The geometric meaning of the gradient as normal to level curves (in 2-D) or level surfaces (in 3-D).

4. Differentials and Linear Approximation

What you should be able to do:

- Compute the total differential $dz = f_x\, dx + f_y\, dy$ for a function $z = (x, y)$.
- Use dz to estimate the change in a function when the inputs change by small amounts.
- Calculate the actual change Δz and compare it to the differential dz.
- Apply a linear (tangent plane) approximation to estimate the value of an expression near a convenient point.

You should understand:

- The differential dz is the linear part of the change in z; it is exact only for linear functions but serves as a close approximation for small changes.
- Δz is the true change in function value; the difference $\Delta z - dz$ measures the error of the linear approximation.
- How to choose a nearby “nice” base point (one with easy arithmetic) when constructing a linear approximation for estimation problems.

5. Critical Points and the Second Derivative Test

What you should be able to do:

- Find critical points by setting both $f_x = 0$ and $f_y = 0$ and solving the resulting system.
- Apply the Second Derivative Test using the discriminant $D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$ to classify each critical point.

You should understand:

- A critical point is a candidate for a local extremum; it is not automatically a maximum or minimum.
- The three outcomes of the Second Derivative Test: $D > 0$ with $f_{xx} > 0$ gives a local minimum; $D > 0$ with $f_{xx} < 0$ gives a local maximum; $D < 0$ gives a saddle point; $D = 0$ is inconclusive.
- Why a saddle point is neither a local max nor a local min—it is a maximum in one direction and a minimum in another.

6. Lagrange Multipliers

What you should be able to do:

- Set up the Lagrange system: $\nabla f = \lambda \nabla g$ along with the constraint $g(x, y) = c$.

• Solve the system of equations to find candidate points and evaluate f at each to identify the optimum.

You should understand:

• Lagrange multipliers are used when you want to optimize a function subject to a constraint—the constraint restricts the domain, so ordinary critical point methods do not apply directly.
• The condition $\nabla f = \lambda \nabla g$ says that at an optimum on the constraint curve, the gradient of f is parallel to the gradient of the constraint function.
• The scalar λ itself is an intermediate quantity; the goal is to find the optimal x and y values and the corresponding function value.

Practice Worksheet

Limits

1. Evaluate the following limit by direct substitution, or explain why it cannot be evaluated that way:

$$\lim_{(x,y)\to(2,-1)} \frac{3x^2y}{\sqrt{x^2+y^2+4}}$$

2. Consider $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+4y^2}$.
   a. Find the limit along $y = 0$.
   b. Find the limit along $x = 0$.
   c. What can you conclude? Explain your reasoning.

Partial Derivatives

3. Given $f(x,y) = 3x^3y^2 - 2x^2y^4 + 5y$, find f_x, f_y, f_{xx}, and f_{xy}.

Directional Derivatives and the Gradient

4. Let $f(x,y) = x^2y - y^3$. At the point $(1,2)$:
   a. Find the gradient of f.
   b. Find the directional derivative in the direction of $\langle 3,-4\rangle$.
   c. In what direction does f increase most rapidly, and what is that maximum rate of change?

5. The temperature in a room (in °C) at position (x,y,z) in meters is given by $T(x,y,z) = 50e^{-2x^2-y^2-3z^2}$. At the point $(1,0,1)$, moving in the positive x-direction, is the temperature increasing or decreasing? Justify your answer.

Differentials and Linear Approximation

6. Given $z = x^3 - 3xy^2$, $f(x,y)$ changes from $(1,2)$ to $(1.03, 1.97)$, find and compare dz and Δz.

7. Use a linear approximation to estimate the value of $\sqrt{15.95} - \sqrt[4]{81.15}$ to 6 decimal places. Choose an appropriate base point and show your setup.

Critical Points

8. Find the critical points of $\boldsymbol{f(x,y) = 2x^3 - 6xy + 3y^2 - 12x}$. Classify each as a local maximum, local minimum, or saddle point using the Second Derivative Test.

Lagrange Multipliers

9. Use the method of Lagrange multipliers to find the minimum value of $f(x,y) = x^2 + 4y^2$ subject to the constraint $x + 2y = 6$.